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1970 J. Phys. A: Gen. Phys. 3 451

(http://iopscience.iop.org/0022-3689/3/5/001)

J. Phys. A: Gen. Phys., 1970, Vol. 3. Printed in Great Britain

When is the 'sum over classical paths' exact?

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MS. received 20*th April* 1970

Abstract. The calculations of Schulman on the path integral formulation of the spherical top are generalized to the case of a free particle moving on the manifold of a simple Lie group. It is shown that the finite time propagator takes on the same form as the short time one except for the phase factor $\exp(-\frac{1}{12}\mathrm{i}Rt)$ where R is the constant scalar curvature of the group manifold.

1. Introduction

The path integral formalism, despite some mathematical imprecision as to its meaning, allows us to formulate quantum mechanics, and other things, in a direct and satisfying way (Feynman and Hibbs 1965).

The quantity that the theory deals with is the probability amplitude, or propagator, $\langle q'', t''|q', t'\rangle$, the square of which gives the probability of finding our system in the state labelled by the numbers (coordinates) q''^i, $i = 1...r$ at time t'' if the system was definitely in the state q'^i at time $t' < t''$. One way of deriving, and thereby defining, the path integral expression for the propagator is the method used by Feynman (1948) based on work of Dirac. The propagator is split into infinitely many pieces by means of the composition law (semi-group property)

$$\langle q''', t'''|q', t'\rangle = \int \langle q''', t'''|q'', t''\rangle \, \mathrm{d}q'' \langle q'', t''|q', t'\rangle \tag{1}$$

(which is simply a statement of the completeness of the states $|q'', t''\rangle$), and the functional integration 'over all paths' is interpreted as an integration over a continuous product of differentials. All we need now is an expression for the 'short time' propagator $\langle q'', t+\Delta t|q', t\rangle$.

Essentially by postulate we take the following expression

$$\langle q'', t'+\Delta t|q', t'\rangle = N\exp\{\mathrm{i}S(q'', t'+\Delta t|q', t')\} \tag{2}$$

where N is some normalization and S is the classical 'action' defined by

$$S(q'', t''|q', t') = \int_{t'}^{t''} \mathrm{L}(q, \dot{q}, \ldots, t)\, \mathrm{d}t \qquad \begin{array}{l} q'' = q(t'') \\ q' = q(t') \end{array}$$

in terms of the classical Lagrangian L. In equation (2), S is calculated for the *classical* path connecting q' and q''. The finite time propagator is now calculated, at least in principle, by substituting (2) into the functional integral, which now takes on the aspect of an integral over all paths connecting the two end points, each path being composed, so to speak, of a set of infinitesimal classical paths. The remarkable thing is that in those cases where the propagator has been calculated in closed form the expression for the *finite* time propagator coincides, more or less, with that for the *short time* one, (2). This fact has been noted by Schulman (1968) and DeWitt (1969).

It means that the integral over *all* paths reduces to a sum over *classical* paths.† That this is always the case has been claimed by Clutton–Brock (1965), but this is incorrect.‡ In this paper we should like to give a class of systems for which the result is true.

2. Calculation. Basic ideas and an explicit example

We shall choose systems whose Lagrangians are at most quadratic in velocities and, amongst these, those with Lagrangians given by

$$\mathrm{L} = \tfrac{1}{2}g_{ij}(q)\,\dot{q}^i\,\dot{q}^j\,.$$

The quantization of such a system has been discussed by DeWitt (1957) and we shall use the notation and results of this paper. The system is just that of a 'particle' restricted to move in an r-dimensional Riemannian space V_r of metric g_{ij}.

To proceed with the quantization we need an expression for the short time propagator. To this end the following structure is introduced (DeWitt 1957—§ 7):

$$\langle q'', t''|q', t'\rangle_c \equiv (2\pi\mathrm{i})^{-r/2}g''^{-1/4}D^{1/2}(q'', t''|q', t')g'^{-1/4}\exp\{\mathrm{i}S(q'', t''|q', t')\} \tag{3}$$

where $g' = \det|g_{ij}(q')|$ etc. and D is the van Vleck determinant

$$D = -\det\left|\frac{\partial^2 S}{\partial q''^j \partial q'^i}\right|. \tag{4}$$

By explicit calculation $\langle q'', t''|q', t'\rangle_c$ is shown to *nearly* satisfy a Schrödinger equation, i.e. we have, for $t'' > t'$,

$$\begin{aligned}
&\mathrm{i}\frac{\partial}{\partial t''}\langle q'', t''|q't'\rangle_c - H_{q''}\langle q'', t''|q', t'\rangle_c \\
&= \tfrac{1}{2}g''^{-1/4}D^{-1/2}\frac{\partial}{\partial q''^i}\left\{g''^{1/2}g''^{ij}\frac{\partial}{\partial q''^j}(g''^{-1/4}D^{1/2})\right\}\langle q'', t''|q', t'\rangle_c \\
&= \{\tfrac{1}{12}R' + \mathrm{o}(q''-q') + \mathrm{o}(t''-t')\}\langle q'', t''|q', t'\rangle_c
\end{aligned} \tag{5}$$

where $R(q)$ is the Riemannian scalar curvature of V_r and where the operator $H_{q''}$ is defined by

$$H_{q''} \equiv -\tfrac{1}{2}g''^{-1/2}\frac{\partial}{\partial q''^i}\left(g''^{1/2}g''^{ij}\frac{\partial}{\partial q''^j}\right)$$

when acting on a wave function. This is just the Laplace–Beltrami operator Δ_2 in the Riemannian space V_r. Because of the R' term on the right-hand side of (5) we have that

$$\langle q'', t''|q', t'\rangle_c = \langle q'', t''|q', t'\rangle_+ + \mathrm{o}(t''-t')^2$$

where $\langle q'', t''|q', t'\rangle_+$ is the propagator for the Hamiltonian $H + \frac{1}{12}R$ rather than that $\langle q'', t''|q', t'\rangle$ for the Hamiltonian H. To obtain the short time propagator for H in

† If, as is usually the case, there is more than one classical path connecting q' and q'' then we obtain the short time propagator with correct boundary conditions by adding all the corresponding expressions (2), cf. Schulman (1968—p. 1556). This is the sum over classical paths referred to. This problem we do not treat here as it concerns properties of the space V_r in the large.

‡ See the work of Groenewold (1956).

the form of (3) we should have started from the Lagrangian $L+\frac{1}{12}R$. In the case that R is constant the effect is simply that of a constant shift in energy, i.e. a phase factor on the short time propagator, with no physical effect.

DeWitt (1957) goes on to implement the Feynman path summation using $\langle|\rangle_c$ as the short time form for $\langle|\rangle_+$ with a rather surprising conclusion which does not concern us here as we are going to ask the following question. Under what circumstances is expression (3)† the *finite* time, or exact, propagator, up to a phase factor? If this happens then we shall say that the sum over classical paths is exact.

To answer this question we need only consider Schrödinger's equation and so we return to a consideration of (5). The condition we are seeking is that the following equation should hold:

$$\tfrac{1}{2}g''^{-1/4}D^{-1/2}\frac{\partial}{\partial q''^i}\left\{g''^{1/2}g''^{ij}\frac{\partial}{\partial q''^j}(g''^{-1/4}D^{1/2})\right\} = \text{constant} \tag{6}$$

i.e. that the quantity $g''^{-1/4}D^{1/2}g'^{-1/4}$ should be an eigenfunction of the Laplace–Beltrami operator. It is easily shown that the van Vleck determinant is given by

$$D = -(t''-t')^{-r}\det\left|\frac{\partial^2\Omega}{\partial q''^j\partial q'^i}\right|$$

where Ω is half the square of the geodesic distance between q'' and q', and we see that

$$g''^{1/2}D^{-1}g'^{1/2} = -(t''-t')^r\rho$$

where ρ is Ruse's invariant (e.g. Schouten 1954—p. 383). Condition (6) now becomes

$$\Delta_2''\rho^{-1/2} = c\rho^{-1/2}. \tag{7}$$

Rather than try to attack this equation directly we shall take a hint from the calculations of Schulman (1968). The system there discussed is the spherical top, for which the space V_r is three-dimensional and of constant curvature—the three-dimensional sphere S^3. For the n-dimensional sphere S^n, Walker (1946) has calculated ρ. Specifically, in the case of S^3, we have

$$\rho^{1/2} = (\sin as/as), \qquad \Omega = \tfrac{1}{2}s^2 \qquad \text{and} \qquad a^2 = \tfrac{1}{6}R \tag{8}$$

and it can be checked that $\rho^{-1/2}$ satisfies (7) with c equal to a^2. Substituting these results into Schrödinger's equation (5) we reproduce Schulman's conclusions directly, without having to solve (5). However this is not our main object.

The spheres S^n are harmonic spaces (e.g. Schouten 1954—p. 381) for which ρ is a function of s only. Equation (7) leads uniquely to the form (8) for harmonic spaces.

The *explicit* verification that $\rho^{-1/2}$ for S^3 is an eigenfunction of the Laplace–Beltrami operator is not very satisfactory and to proceed further we note that S^3 is the group manifold of SU(2) and that the zonal spherical functions on this manifold are just the characters of the irreducible representations of SU(2) divided by the dimension of the representation, namely

$$\Phi_h(\theta) = \frac{\sin(h\theta)}{h\sin\theta} \tag{9}$$

† A referee has suggested the happier term 'quasi-classical one-path amplitude' for expression (3). With the qualifications implied in the previous footnote (†) we shall continue to call it the short-time propagator.

where h is a positive integer ($= 2j+1$ in terms of the 'spin' value j) labelling the representation and θ is a rotation angle proportional to the geodesic distance s. For the elementary analysis we have in mind in the present paper we do not need the advanced theory of spherical functions on homogeneous spaces (e.g. Helgason 1962, Berezin *et al.* 1956, Berezin and Gelfand 1956, Berezin 1957, Cartan 1929, Gelfand 1950). It is sufficient to know that the characters, considered as functions on the group manifold, are eigenfunctions of the Casimir operators, considered as differential operators on the manifold. The proof of this elementary fact can be found, for example, in Racah (1951—p. 33).

For SU(2) there is only one independent Casimir operator and that is just the Laplace–Beltrami operator Δ_2. Thus we have

$$\Delta_2\left\{\frac{\sin(h\theta)}{h\sin\theta}\right\} = \lambda\left\{\frac{\sin(h\theta)}{h\sin\theta}\right\}.$$

By taking the limit $h \to 0$ and comparing with (8) we see that $\rho^{-1/2}$ is indeed an eigenfunction of Δ_2 for S^3. Not only that, for we can see possible generalizations of this result. We shall suggest that V_r is the group manifold of a compact semi-simple Lie group and we then have to show that $\rho^{-1/2}$ is related to the zonal functions

$$\Phi_{(h)} = \frac{\chi_{(h)}}{d_{(h)}} \equiv \frac{\text{character of '}(h)\text{ representation'}}{\text{dimension of '}(h)\text{ representation'}} \tag{10}$$

by some limiting procedure analogous to $h \to 0$ for the SU(2) case. In connection with this latter case we note that for $h = 0$ the dimension of the representation appears to be zero. We have not investigated the exact meaning of the value $h = 0$ ($j = -\frac{1}{2}$!) but it seems to us to correspond to the 'zero' representation in which every group element is mapped onto the zero matrix, of any dimension. The character of the unit element, which normally gives the dimension of the representation, here gives zero.

Weyl (1926, 1939) has computed the characters of the irreducible representations of all semi-simple groups. The particular case of SU(n) can be found in Weyl (1931) and, because of its simplicity and usefulness, we shall consider this particular case explicitly.

The configuration space V_r of our dynamical system is now taken to be the group manifold M of SU(n). This is n^2-1 dimensional, thus $r = n^2-1$. The coordinates q^i are now the coordinates of group space, i.e. they are the parameters of the elements of the (abstract) SU(n) group. The metric g_{ij} is the metric with which the group space of SU(n) can be endowed (Cartan 1927, Schouten 1929, Eisenhart 1933). Explicitly† in terms of *canonical* coordinates q^i

$$g_{ij} = 2\delta_{ia}\delta_j^{\ b}\left(\frac{\cosh Q - 1}{Q^2}\right)^a_{\ b} \tag{11}$$

with

$$|Q|^a_b = q^c c_{cb}^{\ \ a}$$

where‡ $q^a = \delta^a_i q^i$ and the $c_{a\,b}^{\cdot\,\cdot\,c}$ are the structure constants of the Lie algebra of SU(n). From the form for ρ we see that we shall need the quantity $g = \det|g_{ij}|$.

† Many of these constructions are valid for any semi-simple group.

‡ The indices a, b, c etc. are introduced for technical reasons.

This quantity occurs, fundamentally, in the expression for the invariant volume in M

$$dq = \sqrt{g}\prod_{1}^{r} \mathrm{d}q^{i}.$$

In canonical coordinates we have from (11), by diagonalizing $\underset{\sim}{Q}$,

$$\sqrt{g} = \prod \frac{\sinh \frac{1}{2}\alpha}{\frac{1}{2}\alpha}$$

where the product extends over all the eigenvalues α of the matrix $\underset{\sim}{Q}$. The Cartan–Killing classification of Lie groups depends on the distribution of these eigenvalues ('roots'). For SU(n) at least $n-1$ ($=l$, the *rank* of the group) vanish and the rest are imaginary and are equal and opposite in pairs. Thus for these non-zero roots we can put, in terms of the 'angles of rotation' ω_t,

$$\alpha = \mathrm{i}(\omega_t - \omega_s), \qquad t, s = 1, 2, \ldots, n$$

with the normalization and ordering

$$\sum_{1}^{n} \omega_t = 0; \qquad \omega_t > \omega_s, \qquad t < s.$$

Thus

$$\sqrt{g} = \left\{\frac{\prod_{t<s} 2\sin\frac{1}{2}(\omega_t - \omega_s)}{D(\omega_1, \omega_2 \ldots \omega_n)}\right\}^2$$

with $D(\omega_1 \ldots \omega_n)$ the difference function

$$D(\omega_1 \ldots \omega_n) = D(\omega) = \prod_{t<s} (\omega_t - \omega_s).$$

We may now use the fact that M is a group space to simplify our basic problem of investigating (7). We can do this by noting that in group space any point is as 'good' as any other point. The group acts transitively on M and any point can, for example, be transformed into the origin O. Let us do this for the starting point q' and choose O as the origin of our canonical coordinate system† where g_{ij} takes on the form δ_{ij}. Thus we have

$$\det \left| \frac{\partial^2 \Omega}{\partial q''^i \partial q'^j} \right| = (-1)^r$$

because q''^i equals se^i with $e^i e_i = 1$. (The unit vectors e^i select a geodesic through O and s is the distance along this curve to the point q''.)

We now have $\rho^{-1/2} \sim g''^{-1/4}$ where‡

$$g''^{-1/4} = \left\{\prod_{t<s} 2\sin\tfrac{1}{2}(\omega_t - \omega_s)\right\}^{-1} D(\omega) \tag{12}$$

† We shall adhere to this special system throughout this paper.

‡ This is just the expression of Walker (1946) for Ruse's invariant in a symmetric space, in the case this last is the manifold of SU(n).

and equation (7) becomes

$$\Delta_2'' g''^{-1/4} = c g''^{-1/4}. \tag{13}$$

At this point we shall simply write down the zonal functions (10) for SU(n) using Weyl's (1939) notation. We have

$$\Phi_{(l)} = \left\{\frac{\xi(l_1, l_2, \ldots l_n\ \ )}{\xi(n-1, n-2, \ldots 0)}\right\}\left\{\frac{D(n-1, n-2, \ldots 0)}{D(l_1, l_2, \ldots l_n)}\right\}. \tag{14}$$

The first factor gives the character and the second the dimension. The integers $l_1 > l_2 > \ldots > l_n = 0$ label the representations of SU(n), in a way which does not concern us here, and the function ξ is defined by

$$\xi(l_1, \ldots l_n) = \det|\exp(\mathrm{i} l_t \omega_s)|.$$

Another way of writing $\xi(n-1, \ldots, 0)$ is

$$\xi(n-1, \ldots, 0) = D\{\exp(\mathrm{i}\omega_1), \ldots, \exp(\mathrm{i}\omega_n)\} = \prod_{t<s} 2 \sin \tfrac{1}{2}(\omega_t - \omega_s) \tag{15}$$

using $\Sigma \omega_t = 0$.

If we compare (12), (13), (14) and (15) we see that all we have to do is to find a set of values for the l_i such that $\xi(l_1, \ldots l_n)$ is given by

$$\xi(l_1, \ldots l_n) \sim D(\omega)\, D(l).$$

For, in this case, we would have $\Phi_{(l)} \sim g^{-1/4}$ and our desired result follows immediately because

$$\Delta_2 \Phi_{(l)} = \lambda_{(l)} \Phi_{(l)}. \tag{16}$$

The desired values are $l_1 = l_2 = \ldots = l_n = 0$. To see this we employ the device of Weyl (1926 or 1939) for investigating the limit $\omega_t \to 0$ but here applied to vanishing l_t. We put l_t equal to $(n-t)h$ and let h tend to zero. We find

$$\xi(l_1 \ldots l_n) \to (\mathrm{i}h)^{n(n-1)/2} D(\omega)$$

and

$$D(l) \to (h)^{n(n-1)/2} \prod_{t<s} (s-t)$$

whence

$$\Phi_{(0)} = \frac{D(\omega)}{\xi(n-1, \ldots, 0)} \left\{\frac{\prod_{t<s} (s-t)}{(n-1)!}\right\}^{-1} \mathrm{i}^{n(n-1)/2}$$

as required. What we have shown now is that the quantity $\langle q'', t'' | q', t' \rangle_c$ satisfies a Schrödinger equation

$$\mathrm{i} \frac{\partial}{\partial t''} \langle q'', t'' | q', t' \rangle_c - H_{q''} \langle q'', t'' | q', t' \rangle_c = \tfrac{1}{2} \lambda_{(0)} \langle q'', t'' | q', t' \rangle_c.$$

The eigenvalue $\lambda_{(0)}$ is related to the constant scalar curvature R of group space which, for a particular choice of normalization, has the value

$$R = \tfrac{1}{4}(n^2 - 1) = \tfrac{1}{4} r.$$

The value of $\lambda_{(l)}$ is the eigenvalue of the Casimir operator in the $(l_1, \ldots, l_n)$ representation and, for l_i equal to zero, reduces to a quarter of the square of the length of the sum of the positive root vectors (see e.g. Racah 1951). For SU(n) we have

$$\tfrac{1}{2}\lambda_{(0)} = \frac{1}{8\cdot 6}(n+1)(n-1)$$

and so

$$= \frac{1}{12}R.$$

This same result can also be shown to be true for the other groups† in Cartan's list, namely B_l, C_l, D_l and the exceptional groups. Using the notation of Racah (1951) the representation for which the zonal function reduces to $g^{-1/4}$ is that given by setting the vector $\boldsymbol{K}$ equal to zero. In this case the eigenvalue of Casimir's operator is $|\boldsymbol{R}|^2$ where the vector $\boldsymbol{R}$ (not to be confused with R the scalar curvature) is half the sum of the positive roots. A useful discussion of these quantities, with a consistent normalization, can be found in Gourdin (1967). The apparent inconsistency of the sign of the eigenvalue of Casimir's operator with that given by Racah is due to a fugitive factor of i^2.

3. Alternative treatment and further developments

Instead of going via the character to our result we can, if we wish, reverse the above procedure. We sketch this approach here.

Firstly we note that $g^{-1/4}(q)$ is a class function‡ and so depends on only the 'complex distance' of q from the origin or, if we like, on the angles ω_s. In this case the Laplace–Beltrami operator Δ_2 acting on $g^{-1/4}$ reduces to the so-called 'radial part' $\mathring{\Delta}_2$ of the operator (e.g. Berezin 1957) and can be expressed in terms of derivatives with respect to the ω_s. We can proceed in the general case, and not just for the algebra A_l, and write

$$g^{-1/4} = \prod^{+} \frac{\boldsymbol{\alpha}\,.\,\boldsymbol{\omega}}{2\sinh\frac{1}{2}(\boldsymbol{\alpha}\,.\,\boldsymbol{\omega})} \tag{17}$$

where the product is over all *positive* root vectors $\boldsymbol{\alpha}$ and the scalar product $\boldsymbol{\alpha}\,.\,\boldsymbol{\omega}$ is taken with respect to the metric in the Cartan subspace (rootspace). We can take this metric to be the unit one. The quantity J^2 defined by

$$J = \prod^{+} 2\sinh\tfrac{1}{2}(\boldsymbol{\alpha}\,.\,\boldsymbol{\omega})$$

is, up to a constant factor, the weight function in group space after all variables except the ω_s have been integrated out. Then (see e.g. Freudenthal 1954) the Laplace–Beltrami operator acting on a class function f becomes

$$\Delta_2 f = \mathring{\Delta}_2 f = \partial^s\partial_s f + 2\partial_s f\partial^s \ln J$$

where

$$\partial^s = \partial_s = \frac{\partial}{\partial\omega_s}.$$

† This is not surprising since SU(n) $= A_l$ is the unitary restriction of the largest group—that of general linear transformations, $SL(n;\,C)$.

‡ Recall that we are using a special coordinate system. If this is relaxed $g^{-1/4}$ is to be replaced by $\rho^{-1/2}$.

A slight rearrangement yields

$$\mathring{\Delta}_2 f = \frac{1}{J}\partial^s\partial_s(Jf) - f\frac{1}{J}\partial^s\partial_s J.$$

We now note that J is an eigenfunction of $\partial^s\partial_s$ with eigenvalue $-|\boldsymbol{R}|^2$, (Freudenthal 1954, Berezin 1957—§ 2.6),

$$\partial^s\partial_s J = -|\boldsymbol{R}|^2 J, \qquad \boldsymbol{R} = \mathrm{i}\tfrac{1}{2}\sum_{+}\boldsymbol{\alpha}. \tag{18}$$

Hence

$$\mathring{\Delta}_2 f = J^{-1}\partial^s\partial_s(Jf) + |\boldsymbol{R}|^2 f. \tag{19}$$

This agrees with Berezin's expression for $\mathring{\Delta}_2$ with a particular value for his constant c, i.e. one chosen to make $\Delta_2 1$ vanish.† (See Berezin 1957—equation (2.22).)

If we choose $g^{-1/4}$ for f in equation (19) we see from (17) that we need the quantity $\partial^s\partial_s\Pi^+(\boldsymbol{\alpha}\,.\,\boldsymbol{\omega})$. This, however, vanishes because of (18) in the limit of the root vectors becoming infinitesimally small. Thus we have that $g^{-1/4}$ is an eigenfunction of Δ_2,

$$\Delta_2 g^{-1/4} = \mathring{\Delta}_2 g^{-1/4} = |\boldsymbol{R}|^2 g^{-1/4}$$

as required. This result ties in with our earlier development because the quantity $J\chi$, χ being the character, satisfies

$$\partial^s\partial_s(J\chi) = -|\boldsymbol{K}|^2(J\chi)$$

(cf. Freudenthal 1954—equation (8.11)). After dividing this equation by the dimension of the representation it is to be compared with

$$\partial^s\partial_s(Jg^{-1/4}) = 0.$$

It is interesting to employ the operator $\mathring{\Delta}_2$ at an earlier stage. The quantity $\langle q'', t''|q', t'\rangle_c$ is, from expression (3), just a function of the complex distance between the two points q'' and q' in the group manifold. This follows from its invariance under left and right translations, which, in turn, can be obtained from the fact that left and right translations transform geodesics into geodesics. Thus we have

$$\begin{aligned}\langle q'', t''|q', t'\rangle \equiv K(q'', q') = K(q''q'^{-1}, 1) = K(q'^{-1}q'', 1) = K(1, q'q''^{-1})\\ = K(1, q''^{-1}q')\end{aligned}$$

where the symbols q'', q', 1 stand for the abstract group elements. Therefore also

$$K(q''q'^{-1}, 1) = K(\xi q''q'^{-1}\xi^{-1}, 1)$$

where ξ stands for a general group element, and so $K(q'', q')$ depends on only the complex distance between the origin, 1, and the point $q''q'^{-1}$ which is the same as the complex distance between q'' and q'. In consequence we can, without loss of generality, discuss the propagator $\langle q'', t''|1, t'\rangle_c$ which, being a class function, satisfies the Schrödinger equation (5) in the form

$$\mathrm{i}\frac{\partial}{\partial t''}\langle q'', t''|1, t'\rangle_c + \tfrac{1}{2}(\mathring{\Delta}_2'' - |\boldsymbol{R}|^2)\langle q'', t''|1, t'\rangle_c = 0 \qquad t'' > t'$$

† See also Helgason (1964) where this and a number of other important results are derived.

or, in view of (19),

$$\mathrm{i}\frac{\partial}{\partial t''}\langle q'', t''|1, t'\rangle_c+\tfrac{1}{2}J''^{-1}\partial''^s\partial_s''(J''\langle q'', t''|1, t'\rangle_c) = 0.$$

This suggests that we introduce a new quantity $\langle q'', t''|q', t'\rangle_c^0$ defined by

$$\langle q'', t''|q', t'\rangle_c^0 \equiv J''\langle q'', t''|q', t'\rangle_c J'$$

which satisfies

$$\mathrm{i}\frac{\partial}{\partial t''}\langle q'', t''|1, t'\rangle_c^0+\tfrac{1}{2}\partial''^s\partial_s''\langle q'', t''|1, t'\rangle_c^0 = 0$$

i.e. an 'ordinary' Schrödinger equation in the Cartan subspace or, as it is more technically known, the maximal torus. Perhaps this equation, more than any other, displays the fact that quantum mechanics, or any other Markovian process, on group space is not so different from that on ordinary flat space. The main distinction is that in group space the relative position of two points is described by l numbers, i.e. the complex distance between the points, where l is the rank of the group.

For a rank l group there are l independent Laplace operators corresponding to the generalizations of the Casimir operator (Racah 1951) and the zonal functions are eigenfunctions of all of them. We have not found it necessary for present purposes to introduce these operators.

Biedenharn (1963) has given a detailed discussion of these quantities and their eigenvalues can be found in Micu (1964) and Wenger (1967) for the case of SU(n).

There should be no difficulty in extending the calculation to the relativistic situation using, for example, a proper-time formulation or to the case of field theory, which latter would be especially interesting in view of its relevance for the chiral symmetry problem.

4. Conclusions

We have shown for a system which is essentially a free point moving in a space M diffeomorphic to the group space of a compact simple group that the Feynman sum over all paths reduces to the sum over classical paths in so far as the finite time propagator is the same as the short time one except for the physically non-significant phase factor $\exp\{-(\mathrm{i}/12)Rt\}$.† This result also applies to the spaces dual to the above in the sense of Cartan (see, e.g. Hermann 1966) and also to direct products of the spaces. This latter corresponds to a trivial compounding of independent systems. Non-compact spaces are also allowed, at least mathematically, as are, of course, spaces of the flat Euclidean kind, for which the g_{ij} can be made constant. There is the remaining question of what is the biggest class of spaces for which the result is true. It cannot be true for all symmetric spaces, for example, because it does not hold for the n-sphere,

$$S^n = \frac{SO(n+1)}{SO(n)} \qquad n > 3.$$

† This result had, in fact, already been conjectured for any Lie group by Schulman in his thesis (1967—Princeton).

We were led to our result in an essentially synthetic way and the calculations were performed explicitly. It seems to us that we could have made more use of the general properties of spherical functions, for example the functional relation,

$$\int \Phi(\eta\xi\eta^{-1}\zeta)\,\mathrm{d}\eta = \Phi(\xi)\,\Phi(\zeta)$$

which equation actually implies that the Φ are given by (10). Also, granted at the outset that the space V_r of the system were a group manifold, the quantization could, no doubt, have been treated in a more direct way using the so-called 'polar coordinates' (Berezin 1957—§ 2.4), i.e. the ω_s and the remaining parameters, from the start.

Finally, we have laid aside all topological considerations as forming a separate chapter. There is, however, just one point we should like to raise at this time. According to Schulman (1968) if there is more than one homotopy class of classical paths, these enter in the Feynman sum with *undetermined* relative phase factors. Now, a similar situation occurs in the Aharonov–Bohm effect where a beam of electrons is made, essentially, to circulate an impenetrable solenoid. Some attempts were made to explain the effect on the basis that the impenetrable solenoid made space multiply-connected and hence one could alter the single-valuedness of the wavefunction. This approach was effectively dismissed by Aharonov and Bohm who simply turned on the solenoid barrier adiabatically. Such a process cannot affect single-valuedness.

Perhaps a similar argument can be applied to the present situation. The fixed axis rigid rotator, a case discussed by Schulman, can be considered to be the limit of a circular potential well, and there is then only one homotopy class of classical paths. This is tantamount to saying that all geometrical constraints are due to forces which have been, or can be, switched on adiabatically. Whether this is a relevant viewpoint is open to discussion.

Acknowledgments

I would like to thank one of the referees for the reference to Groenewold (1956).

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